Ae. E. THESIS

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OPTIMUM THRUST PROGRAMMING FOR LOW THRUST DEVICES

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ABSTRACT

The use of low thrust devices with continuous and discontinuous thrust programs is investigated to determine whether or not a discontinuous thrust program will provide a greater payload and structure mass fraction, where the program must impart a specified amount of energy to the satellite in a specified total time, starting from a circular parking orbit. A discontinuous thrust program is developed, based on the elliptic orbit, using a perturbation analysis, and a series solution is obtained which permits investigation of this program up to angles of one radian either side of the perigee of the elliptic orbit. These results are compared with a continuous thrust program which gave a spiral orbit. Under certain conditions, where storage batteries must be carried as part of the payload but are available for use during the thrust program, the discontinuous thrust program is found to provide a greater payload and structure mass fraction. Further investigation at angles greater than one radian either side of perigee appears to be warranted.

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LIST OF SYMBOLS

g_o	gravity at surface of central mass
R_o	radius at surface of central mass
r	radius at any point in the orbit
θ	angle of position vector in the plane of motion
β	angle of velocity vector in the plane of motion
ϕ	angle of thrust vector in the plane of motion
t, T	time
τ	non-dimensional time
u	non-dimensional reciprocal radius
h	non-dimensional angular momentum
E	energy per unit mass
e	eccentricity
a	acceleration due to thrust
ε	acceleration ratio
m, M	mass
M_o	initial mass of satellite
M_w	mass of power supply
M_p	mass of propellant
M_s	mass of structure
M_l	mass of payload
M_g	mass of electrical generating unit
$M_{batt.}$	mass of batteries
$K_{batt.}$	specific mass of batteries
K	specific mass of power supply

γ	ratio of M_p to M_o
c	exhaust velocity
P	power supplied
$(\bar{\ })$	vector notation
$(\ )'$	derivative with respect to θ
$(\dot{\ })$	derivative with respect to time
PR	perigee restriction
λ	Lagrange multiplier
η	specific power
k	initial cutoff angle
N	number of thrust cycles
FR_{γ^2}	factored ratio of γ^2
μ	ratio of $M_{batt.}$ to M_g
α	energy storage factor

I. INTRODUCTION AND SUMMARY

Once in a parking orbit, low thrust devices may be used to alter the path of a satellite. In particular, it is anticipated that one use of the low thrust device will be to add energy to the satellite for an escape or a near escape condition. The exact path and thrust program must be determined by the mission requirement, but in any event, the system should be optimized for efficient launching.

The problem of optimizing an arbitrary mission is a formidable task to say the least. To simplify matters, it is assumed that the mission is to impart a specified energy in a specified total time to the maximum fraction of payload and structure mass. To do this, one must make efficient use of the power supply and propellant.

The usual variation procedures may be used to derive the differential equations for optimum direction and magnitude of thrust when continuous thrust is employed, as shown by Irving in reference (1). He has shown that a constant acceleration is optimum for the gravity-free case. No analytic solution has been found for the differential equations in the central force field, but Irving has made a numerical study and found the acceleration to be practically constant for various escape programs.

In reference (2), Casey has shown the distinct advantage of energy addition by thrusting near the perigee of an elliptic orbit in such a manner that the perigee remains constant. His thrust program is formulated to require that the perigee distance remain constant throughout the thrust cycle, and results are obtained by

numerical procedures. The reason for thrusting near the perigee is to take advantage of the greater velocity, since in any elliptic orbit the velocity is greatest at the perigee and since the energy addition rate is proportional to the velocity. Further, if the perigee distance is kept constant, then the perigee velocity will increase as the eccentricity of the elliptic orbit increases.

As pointed out by Casey, there will be a period of coasting between each thrust cycle which will make a considerable contribution to the total time. For this reason, we cannot say simply by inspection whether the continuous or discontinuous method of energy addition will provide a greater fraction of payload and structure mass, where both methods are required to impart a specified energy in a specified total time.

To investigate this problem, we shall use the idea of restricting the perigee distance, but only at the beginning and end of each thrust cycle. This permits added freedom in an effort to gain more energy. Since we are only considering low thrust devices, a perturbation technique is used with a simple variation to develop the first perturbation integral equations for the maximum change of energy during one thrust cycle and for the Lagrange multiplier necessary to satisfy the above perigee restriction. The equations apply for any period of thrusting up to one complete circuit of 2π radians, but could not be integrated in closed form. By making a series expansion of the integrands, the equations may be integrated term by term to obtain results useful to an angle of one radian either side of

the perigee. While this does not permit a complete comparison of these two methods of energy addition, it is possible to draw definite conclusions for angles less than one radian.

After obtaining equations for one cycle of energy addition, the effect of a large number of cycles is obtained by integrating the changes of the various elliptic orbit parameters.

To compare these results, an approximate solution for the spiral orbit, using a low thrust device, is obtained. Then a comparison is made of the payload and structure mass fraction, where both methods are required to fulfill the mission of imparting a specified energy in a specified total time.

Finally, since the discontinuous thrust program could generate energy while coasting, the use of storage batteries to augment the power available during the thrust cycle will be considered. In one case, the batteries will be considered as part of the power supply; and in another, they will be taken as part of the payload, but available for use during the thrusting program.

When thrust is only permitted up to an angle of one radian either side of perigee, the discontinuous thrust program was found to have no payload and structure mass fraction advantage unless batteries are employed. When the silver-zinc battery is considered as part of the power supply, there is an area of marginal advantage due to the low specific mass of this battery, but its poor recharging reliability rules out its present use in such a system.

The nickel-cadmium storage battery, which is very reliable, provided an area of definite advantage for the discontinuous

thrust program when the battery is considered as part of the payload.

II. PRELIMINARY ANALYSIS

We shall first write the equations of motion for the satellite, making the usual simplifying assumptions of a very large, central, spherical mass and take the coordinate system at the center. We shall further assume motion in a plane, keeping the thrust in the plane of the orbit, and neglect the effects of other bodies. Then, using plane polar coordinates, as in fig. 1, and the dot notation for time derivative, we have:

$$\bar{r}_m = \bar{i}\, r \tag{2.1}$$

$$\dot{\bar{r}}_m = \bar{i}\,\dot{r} + \bar{j}\, r\dot{\theta} \tag{2.2}$$

$$\ddot{\bar{r}}_m = \bar{i}\,(\ddot{r} - r\dot{\theta}^2) + \bar{j}\left(\frac{1}{r}\frac{d}{dt}\{r^2\dot{\theta}\}\right) \tag{2.3}$$

Equating equation 2.3 to the acceleration due to thrust, $\bar{a}$, and gravity, we have:

$$\bar{i}\,(\ddot{r} - r\dot{\theta}^2) + \bar{j}\left(\frac{1}{r}\frac{d}{dt}\{r^2\dot{\theta}\}\right) = \bar{i}\left(a\cos(\beta+\phi) - \frac{g_o R_o^2}{r^2}\right) + \bar{j}\,(a\sin(\beta+\phi)) \tag{2.4}$$

where R_o is the radius of the central mass, and g_o is the acceleration of gravity at $r = R_o$. Since a/g_o is small for a low thrust device, we shall first take $a = 0$ and obtain the usual solution in dimensionless form. The thrust will be considered later as a perturbation. Choosing θ as the independent variable, and letting ( )' be differentiation with respect to θ, then with U as the dependent

variable and h as the dimensionless angular momentum, we have:

$$u = \frac{R_o}{r} \tag{2.5}$$

$$r^2\dot{\theta} = \sqrt{g_o/R_o}\, R_o^2\, h \tag{2.6}$$

but when $a = 0$, $\frac{d}{dt}(r^2\dot{\theta}) = 0$; therefore h = constant = h_o .

$$\dot{\theta} = \sqrt{g_o/R_o}\, h_o\, u^2 \tag{2.7}$$

$$\dot{r} = -\sqrt{g_o R_o}\, h_o\, u' \tag{2.8}$$

$$\ddot{r} = -g_o h_o^2 u^2 u'' \tag{2.9}$$

$$r\dot{\theta}^2 = g_o h_o^2 u^3 \tag{2.10}$$

Substituting into equation 2.4, taking $a = 0$, and simplifying, we have:

$$u'' + u = \frac{1}{h_o^2} \tag{2.11}$$

A solution may be written

$$u = u_o = \frac{1}{h_o^2}\left[1 + e_o \cos(\theta - \theta_p)\right] \tag{2.12}$$

where e_o is the eccentricity and θ_p is the angle of the perigee position. The non-dimensional time, $\Delta\tau$, for any portion of the orbit is given by:

$$\Delta\tau = \sqrt{\frac{g_o}{R_o}}(T_2 - T_1) = \sqrt{\frac{g_o}{R_o}}\int_{\theta_1}^{\theta_2} dt = \sqrt{\frac{g_o}{R_o}}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\dot{\theta}} = h_o^3\int_{\theta_1}^{\theta_2}\frac{d\theta}{(1+e_o\cos\theta)^2} \qquad (2.13)$$

using equations 2.7 and 2.12. The integral can be evaluated to give:

$$\Delta\tau = \frac{2h_o^3}{(1-e_o^2)^{3/2}}\left\{\tan^{-1}\left(\sqrt{\frac{1-e_o}{1+e_o}}\tan\frac{\theta}{2}\right) - \frac{e_o\sqrt{1-e_o^2}\tan\frac{\theta}{2}}{(1+e_o)+(1-e_o)\tan^2\frac{\theta}{2}}\right\}\Bigg]_{\theta_1}^{\theta_2}$$

In equations 2.14 through 2.26, we shall essentially follow the work of Irving, reference (1), since his approach provides a convenient parameter for computing the mass fraction of the satellite.

The mass of the power supply, M_W , is assumed to be proportional to the power supplied, P :

$$M_W = KP \qquad (2.14)$$

where K is the specific mass of the power supply. It is further assumed that the maximum power is utilized when thrusting, appearing as kinetic energy in the exhaust:

$$P = \frac{1}{2}(-\dot{m})c^2 \qquad (2.15)$$

where c is the exhaust velocity, m the mass of the vehicle at time t , and $-\dot{m}$ is the propellant flow rate. We also have the acceleration of the satellite due to reaction:

$$\bar{a} = \frac{\overline{\text{thrust}}}{\text{mass}} = \frac{\dot{m}\bar{c}}{m} \qquad (2.16)$$

Using equations 2.15 and 2.16, we eliminate c :

$$\frac{\bar{a}\cdot\bar{a}}{2P} = \frac{\dot{m}^2[\bar{c}\cdot\bar{c}]/m^2}{-\dot{m}c^2} = -\frac{\dot{m}}{m^2} = \frac{d}{dt}\left(\frac{1}{m}\right) \tag{2.17}$$

Equation 2.17 may be integrated to obtain an expression for mass as a function of time:

$$\frac{1}{M} = \frac{1}{M_o} + \frac{1}{2}\int_0^t \frac{a^2}{P}\,dt \tag{2.18}$$

where M_o is the initial mass of the satellite. If we let M_p be the propellant mass, then the "burn out" mass, M_b, at time T_b is given by:

$$M_b = M_o - M_p \tag{2.19}$$

Substituting in equation 2.18 and using 2.14, we have:

$$\frac{M_o}{M_b} = 1 + \frac{K}{2}\frac{M_o}{M_w}\int_0^{T_b} a^2\,dt \tag{2.20}$$

To optimize the power supply, we write the relation between the payload mass, M_1, the structure mass, M_s, and the other masses:

$$M_1 + M_s + M_p + M_w = M_o \tag{2.21}$$

Further, let

$$\gamma^2 = \frac{K}{2}\int_0^{T_b} a^2\,dt \tag{2.22}$$

Then, manipulating equations 2.19, 2.20, 2.21, and 2.22, we have:

$$\frac{M_1 + M_s}{M_o} = \frac{M_w}{M_o}\left[\frac{1}{(M_w/M_o) + \gamma^2} - 1\right] \tag{2.23}$$

There is an infinite number of programs for $\bar{a}$ which will keep γ^2 constant while fulfilling the mission as M_w/M_0 is varied. Thus, taking the derivative of equation 2.23 equal to zero, we have the maximum of $(M_1 + M_s)/M_0$ with respect to M_w/M_0 when:

$$\frac{M_w}{M_0} = \gamma - \gamma^2 \tag{2.24}$$

Substituting equation 2.24 into 2.23, we have:

$$\frac{M_1 + M_s}{M_0} = (1 - \gamma)^2 \tag{2.25}$$

Using equations 2.21, 2.24, and 2.25, we have:

$$\frac{M_p}{M_0} = \gamma \tag{2.26}$$

From equation 2.26 we see the range of γ must be zero to one, since M_p/M_0 must lie in this range. Further, for a particular mission, we see from equation 2.25 that the structure and payload mass fraction will be greatest when γ is a minimum. After determining the minimum value for γ, the optimum power supply mass is given by equation 2.24.

We are now ready to investigate equation 2.22. In reference (1), Irving has shown analytically that the acceleration must vary linearly with time in gravity-free space, if γ is to be a minimum. Further, when the mission of specified energy in a specified time is considered, the optimum acceleration becomes constant. For the case of a central force field, Irving made a numerical study using continuous thrust, and found the optimum acceleration to be practically constant. Thus, the spiral orbit with constant acceleration is a

"near optimum" solution which will provide a fair comparison between continuous and discontinuous thrust programming.

If we consider a discontinuous thrust program applied to an elliptic orbit, it seems clear that the thrust should be applied near the perigee, where the velocity is greatest, to impart the greatest energy. If the thrust is parallel to the velocity vector, we will impart the maximum energy during any given thrust cycle. Such a program will allow the perigee distance to increase, and with this increase there is a decrease of perigee velocity, thus decreasing the energy imparted during a subsequent given thrust period. To obtain the advantage of thrusting near the perigee, but still keeping the same distance for a velocity advantage in subsequent cycles, a program must be found that imparts maximum energy during any given thrust cycle while providing no net change in the perigee distance. After obtaining the restricted perigee equations for one cycle, the programming of thrust time will be considered and γ will be computed for a large number of cycles.

After developing the restricted perigee equations, we shall obtain an approximate solution for the spiral orbit with continuous thrust to compare with the restricted perigee thrust program. Finally, we shall consider the use of batteries with the restricted perigee thrust program. The batteries can be charged while coasting to provide added power when thrusting. In one case, the batteries will be considered as part of the power supply, and in another case they will be considered as part of the payload.

III. PERTURBATION EQUATIONS

When $\bar{a}$ is not zero, we have a small non-dimensional acceleration:

$$\epsilon = \frac{a}{g_0} << 1 \tag{3.1}$$

since we are considering only low thrust devices. This suggests the use of a perturbation analysis for each cycle.

Since h is no longer constant, we must rewrite equation 2.9:

$$\ddot{r} = -g_0 h u^2 (h'u' + h u'') \tag{3.2}$$

and for the $\bar{j}$ component of equation 2.3 we also have:

$$\frac{1}{r} \frac{d}{dt}(r^2 \dot{\theta}) = g_0 u^3 h h' \tag{3.3}$$

By using fig. 1 and simple trigonometry, we write:

$$\sin(\phi + \beta) = \frac{-u' \sin\phi + u \cos\phi}{\sqrt{u^2 + u'^2}} \tag{3.4}$$

and

$$\cos(\phi + \beta) = \frac{u' \cos\phi - u \sin\phi}{\sqrt{u^2 + u'^2}} \tag{3.5}$$

Substituting equations 2.5, 2.10, 3.1, 3.2, 3.3, 3.4, and 3.5 into equation 2.4, we have the radial equation:

$$u'' + u + \frac{h'u'}{h} = \frac{1}{h^2} + \epsilon \left(\frac{u' \cos\phi + u \sin\phi}{h^2 u^2 \sqrt{u^2 + u'^2}} \right) \tag{3.6}$$

and the tangential equation

$$h' = \epsilon\left(\frac{-u'\sin\phi + u\cos\phi}{u^3 h\sqrt{u^2+u'^2}}\right) \tag{3.7}$$

We get some simplification in equation 3.6 by substituting for h' from equation 3.7:

$$u'' + u = \frac{1}{h^2} + \epsilon\,\frac{\sqrt{u^2+u'^2}}{h^2 u^3}\sin\phi \tag{3.8}$$

For the perturbation solution, we take h_o constant and u_o as given by equation 2.12 to write the assumed solutions:

$$u = u_o + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \tag{3.9}$$

and

$$h = h_o + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + \cdots \tag{3.10}$$

where $u_1 = u_2 = \cdots = h_1 = h_2 = \cdots = 0$ at θ_1, the angle where the thrust for a particular cycle is started. Substituting equations 3.9 and 3.10 into equations 3.7 and 3.8, we obtain the first perturbation differential equations:

$$h_1' = \frac{-u_o'\sin\phi + u_o\cos\phi}{h_o u_o^3\sqrt{u_o^2+u_o'^2}} \tag{3.11}$$

and

$$u_1'' + u_1 = -\frac{2h_1}{h_o^3} + \frac{\sqrt{u_o^2+u_o'^2}}{h_o^2 u_o^3}\sin\phi \tag{3.12}$$

For a particular cycle, where the thrust is started at angle θ_1 and $\theta - \theta_1 \leq 2\pi$, the solutions may be written:

$$h_1(\theta) = \int_{\theta_1}^{\theta} h_1'(\psi)\,d\psi \tag{3.13}$$

and

$$u_1(\theta) = \int_{\theta_1}^{\theta} \sin(\theta-\psi)\left[-\frac{2}{h_o^3}\int_{\theta_1}^{\psi} h_1'(x)\,dx + \frac{\sqrt{u_o^2+u_o'^2}}{h_o^2\, u_o^3}\sin\phi\right] d\psi \qquad (3.14)$$

For a unit mass, the energy is given by

$$\begin{aligned} E &= \text{kinetic energy} + \text{potential energy} \\ &= \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + g_o R_o^2 \int_{R_o}^{r} \frac{dr}{r^2} \\ &= \frac{g_o R_o}{2}(h^2 u'^2 + h^2 u^2) + g_o R_o (1-u) \\ &= g_o R_o \left[1 - \frac{1-e^2}{2h^2} \right] \end{aligned} \qquad (3.15)$$

by using equations 2.5, 2.6, 2.7, 2.8, and 2.12, and where the potential energy is considered zero at $r = R_o$. By substituting equations 3.9 and 3.10 into equation 3.15, we obtain the first perturbation energy change:

$$\frac{\Delta E}{\epsilon g_o R_o} = h_o(u_o^2 + u_o'^2)h_1 + (h_o^2 u_o - 1)u_1 + h_o^2 u_o' u_1' \qquad (3.16)$$

The perigee distance ratio is obtained by evaluating equation 2.12 at $\theta = \theta_p$:

$$\frac{R_o}{r_p} = \frac{1+e_o}{h_o^2} \qquad (3.17)$$

We could require that this quantity remain constant during the thrust cycle, as done by Casey in reference (2), but we should have a gain in the energy imparted if we require that the perigee distance ratio be equal before and after the cycle. We shall pursue this latter method of perigee restriction and will demonstrate a slight gain of energy over the method of continuous perigee restriction.

The eccentricity may be evaluated before and after the thrust cycle:

$$e = \sqrt{(u h^2 - 1)^2 + u'^2 h^4} \tag{3.18}$$

by using equation 2.12. Then, substituting equations 3.9, 3.10, and 3.18 into equation 3.17 and equating the results before and after the thrust cycle, we obtain the first perturbation perigee restriction:

$$PR = 0 = 2\left[h_0 u_0^2 + h_0 u_0'^2 - \frac{u_0}{h_0} - \frac{e_0(1+e_0)}{h_0^3}\right] h_1 + \\ + \left[h_0^2 u_0 - 1\right] u_1 + h_0^2 u_0' u_1' \tag{3.19}$$

evaluated at θ_2, where θ_2 is the angle at the end of the thrust cycle.

IV. OPTIMIZATION OF THE RESTRICTED PERIGEE CYCLE

To find the thrust angle which will impart maximum energy during the thrust cycle and still satisfy the perigee restriction given by equation 3.19, we shall take a simple variation, using the Lagrange multiplier, equal to zero. We write the non-dimensional energy, equation 3.16, plus the perigee restriction constraint, equation 3.19, with the Lagrange multiplier, λ , where λ is a constant to be determined, and, assuming ϵ = constant during the cycle, take a variation on ϕ equal to zero:

$$\delta\left[\frac{\Delta E}{\epsilon g_o R_o} + \lambda PR\right] = 0 \tag{4.1}$$

For convenient notation, we rewrite equations 3.16 and 3.19 as

$$\frac{\Delta E}{\epsilon g_o R_o} = A h_1 + B u_1 + C u_1' \tag{4.2}$$

and

$$PR = 0 = A' h_1 + B u_1 + C u_1' \tag{4.3}$$

where A , A' , B , and C are shown in equations 3.16 and 3.19. Here, we note,by taking the difference of equations 4.2 and 4.3, that

$$\frac{\Delta E}{\epsilon g_o R_o} = (A - A')\, h_1(\theta_2) \tag{4.4}$$

is the energy change for the thrust cycle. Substituting equations 4.2 and 4.3 into 4.1, we have:

$$(A + \lambda A')\delta h_1 + (1+\lambda)\left[B\delta u_1 + C\delta u_1'\right] = 0 \tag{4.5}$$

Before taking the variation, we must first integrate equation 3.14 by parts to obtain the more useful form:

$$u_1(\theta) = \int_{\theta_1}^{\theta} \sin(\theta-\psi)\frac{\sqrt{u_0^2+u_0'^2}}{h_0^2\, u_0^3}\sin\phi\, d\psi + \frac{2}{h_0^3}\int_{\theta_1}^{\theta}\left[\cos(\theta-\psi)-1\right] h_1'(\psi)\, d\psi \tag{4.6}$$

Then, taking a derivative of equation 4.6 with respect to θ :

$$u_1'(\theta) = \int_{\theta_1}^{\theta} \cos(\theta-\psi)\frac{\sqrt{u_0^2+u_0'^2}}{h_0^2\, u_0^3}\sin\phi\, d\psi - \frac{2}{h_0^3}\int_{\theta_1}^{\theta}\sin(\theta-\psi)\, h_1'(\psi)\, d\psi \tag{4.7}$$

Using equation 3.11, we note that:

$$\delta h_1(\theta_2) = \int_{\theta_1}^{\theta_2} \delta h_1'(\psi)\, d\psi = \int_{\theta_1}^{\theta_2} \frac{-u_0'\cos\phi - u_0\sin\phi}{h_0\, u_0^3\sqrt{u_0^2+u_0'^2}}\,\delta\phi\, d\psi \tag{4.8}$$

Now, taking the variation of equations 4.6 and 4.7, using 4.8, we substitute into equation 4.5 to obtain:

$$\delta\left[\frac{\Delta E}{\epsilon g_0 R_0} + \lambda PR\right] = \int_{\theta_1}^{\theta_2} \frac{(1+\lambda)}{h_0\, u_0^3\sqrt{u_0^2+u_0'^2}}\left\{\frac{u_0^2+u_0'^2}{h_0}\Big(B\sin(\theta_2-\psi) + C\cos(\theta_2-\psi)\Big)\cos\phi \, - \right.$$

$$\left. - \left(\frac{A+\lambda A'}{1+\lambda} + \frac{2B}{h_0^3}\left[\cos(\theta_2-\psi)-1\right] - \frac{2C}{h_0^3}\sin(\theta_2-\psi)\right)\Big(u_0\sin\phi + u_0'\cos\phi\Big)\right\}\delta\phi\, d\psi$$

$$\tag{4.9}$$

For equation 4.9 to be zero, we must require that the integrand be zero, so that

$$U_0 \sin\phi\left[\frac{A+\lambda A'}{1+\lambda} + \frac{2B}{h_0^3}(\cos(\theta_2-\psi)-1) - \frac{2C}{h_0^3}\sin(\theta_2-\psi)\right] =$$

$$= -U_0'\cos\phi\left[\frac{A+\lambda A'}{1+\lambda} + \frac{2B}{h_0^3}(\cos(\theta_2-\psi)-1) - \frac{2C}{h_0^3}\sin(\theta_2-\psi) - \right.$$

$$\left. - \frac{U_0^2+U_0'^2}{h_0 U_0'}\left(B\sin(\theta_2-\psi) + C\cos(\theta_2-\psi)\right)\right] \qquad (4.10)$$

Substituting the expressions for A , A' , B , and C from equations 3.16, 3.19, 4.2, and 4.3, using equation 2.12, and taking $\theta_p = 0$, equation 4.10 becomes

$$\tan\phi = -\frac{\sin\theta}{1+e_0\cos\theta}\left[\frac{\lambda e_0(1+e_0)^2}{1+e_0^2+2e_0\cos\theta+2\lambda e_0(\cos\theta-1)}\right] \qquad (4.11)$$

where λ is a constant to be determined. It will be shown that the shift of θ_p is negligible for the symmetrical thrust cycle, so that a choice of zero for θ_p presents no problem in going from one thrust cycle to the next. We now have the thrust direction program given by equation 4.11.

Substituting the expressions for A and A' and equations 2.12, 3.11, and 4.11 into 4.4, we have:

$$\frac{\Delta E}{\epsilon g_0 R_0 h_0^2} =$$

$$= \int_{\theta_1}^{\theta_2} \frac{(1+e_0)^2(1+e_0\cos\psi)^2\left[1+e_0^2+2e_0\cos\psi+2\lambda e_0(\cos\psi-1)\right] - e_0(1+e_0)^4\lambda e_0\sin^2\psi}{(1+e_0\cos\psi)^3\sqrt{1+e_0^2+2e_0\cos\psi}\sqrt{(1+e_0\cos\psi)^2\left[1+e_0^2+2e_0\cos\psi+2\lambda e_0(\cos\psi-1)\right]^2 + \lambda^2 e_0^2(1+e_0)^4\sin^2\psi}}\, d\psi$$

(4.12)

Also, substituting equations 3.11, 4.6, 4.7, and 4.11 into 3.19, and

using 2.12, we have:

$$PR = 0 =$$

$$= \int_{\theta_1}^{\theta_2} \frac{\lambda e_o (1+e_o)^4 \sin^2\psi + 2(1+e_o\cos\psi)^2\left[1+e_o^2+2e_o\cos\psi+2\lambda e_o(\cos\psi-1)\right](\cos\psi-1)}{(1+e_o\cos\psi)^3\sqrt{1+e_o^2+2e_o\cos\psi}\sqrt{(1+e_o\cos\psi)^2\left[1+e_o^2+2e_o\cos\psi+2\lambda e_o(\cos\psi-1)\right]^2 + \lambda^2 e_o^2(1+e_o)^4\sin^2\psi}}\, d\psi \tag{4.13}$$

From equation 4.13 we determine the value of the constant λe_o for any particular cycle, since λ always occurs with e_o in the two equations 4.12 and 4.13.

Since it is not possible to integrate the above equations exactly, we must use numerical methods to carry the investigation to large angles. This would be fairly simple for one cycle, but when one considers a large number of cycles, a computer program appears to be mandatory. We shall not pursue this route, but instead, we shall present an approximate evaluation which is useful to angles of one radian.

V. APPROXIMATE EVALUATION FOR ONE CYCLE

To make an approximate evaluation of the optimum restricted perigee equations for one cycle, we shall make series expansions and integrate term by term. First, using equation 4.13, we may rewrite 4.12 as:

$$\frac{\Delta E}{\epsilon g_o R_o h_o^2} = -e_o PR + \int_{\theta_1}^{\theta_2} \frac{(1+e_o^2+2e_o\cos\psi)(1+e_o\cos\psi)^2[1+e_o^2+2e_o\cos\psi+2\lambda e_o(\cos\psi-1)]}{\text{DENOMINATOR}}\, d\psi \tag{5.1}$$

since $PR = 0$ by the appropriate choice of λe_o and where DENOMINATOR is the denominator of the integrand in both equations 4.12 and 4.13.

Now consider the function

$$G = \int_{\theta_1}^{\theta_2} \Phi \, d\psi \tag{5.2}$$

where

$$\Phi = \frac{\sqrt{(1+e_o\cos\psi)^2[(1+e_o^2+2e_o\cos\psi)b+2\lambda e_o(\cos\psi-1)]^2+\lambda^2 e_o^2(1+e_o)^4\sin^2\psi}}{(1+e_o\cos\psi)^3\sqrt{1+e_o^2+2e_o\cos\psi}} \tag{5.3}$$

We notice that

$$\frac{1}{e_o}\frac{\partial G}{\partial \lambda}\bigg|_{b=1} = \frac{1}{e_o}\frac{\partial}{\partial\lambda}\int_{\theta_1}^{\theta_2}\Phi\, d\psi\bigg|_{b=1} = \frac{1}{e_o}\int_{\theta_1}^{\theta_2}\frac{\partial\Phi}{\partial\lambda}\, d\psi\bigg|_{b=1} \tag{5.4}$$

has an integrand identical with equation 4.13, and

$$\frac{\partial G}{\partial b}\bigg|_{b=1} = \frac{\partial}{\partial b}\int_{\theta_1}^{\theta_2}\Phi\, d\psi\bigg|_{b=1} = \int_{\theta_1}^{\theta_2}\frac{\partial\Phi}{\partial b}\, d\psi\bigg|_{b=1} \tag{5.5}$$

has an integrand identical with the non-vanishing term of equation 5.1, since $PR = 0$. Thus, if we perform the integration of equation 5.2, we may evaluate λe_o and the increase in energy by taking the derivatives indicated in equations 5.4 and 5.5.

Expanding in series, we collect the coefficients of the various terms of Φ and integrate. Noting that the result is symmetrical about $\theta = 0$, we write

$$\frac{1}{2} G = \frac{b}{(1+e_o)} \left\{ \theta_2 + \frac{N_1}{(1+e_o)^2} \frac{\theta_2^3}{6} + \frac{N_2}{(1+e_o)^4} \frac{\theta_2^5}{120} + \frac{N_3}{(1+e_o)^6} \frac{\theta_2^7}{10,080} + \cdots \right\} \tag{5.6}$$

where

$$N_1 = 2 e_o^2 + e_o - 2\sigma + \sigma^2 \tag{5.7}$$

$$\begin{aligned} N_2 = {} & 16 e_o^4 + 19 e_o^3 - e_o^2 - e_o - 22\sigma e_o^2 - 32\sigma e_o + 2\sigma + 20\sigma^2 e_o^2 + \\ & + 34\sigma^2 e_o - 4\sigma^2 + 12\sigma^3 - 3\sigma^4 \end{aligned} \tag{5.8}$$

$$\begin{aligned} N_3 = {} & 544 e_o^6 + 1038 e_o^5 + 242 e_o^4 - 362 e_o^3 - 18 e_o^2 + 2 e_o - 349\sigma e_o^4 - 2416\sigma e_o^3 - \\ & - 1746\sigma e_o^2 + 344\sigma e_o - 4\sigma + 1232\sigma^2 e_o^4 + 3638\sigma^2 e_o^3 + 2802\sigma^2 e_o^2 - \\ & - 922\sigma^2 e_o + 32\sigma^2 + 1140\sigma^3 e_o^2 + 2640\sigma^3 e_o - 300\sigma^3 - 420\sigma^4 e_o^2 - \\ & - 930\sigma^4 e_o + 840\sigma^4 - 540\sigma^5 + 90\sigma^6 \end{aligned} \tag{5.9}$$

and for convenience we have used

$$\sigma = \frac{\lambda e_o}{b} \tag{5.10}$$

Now, taking the derivative of equation 5.6 as indicated in equation 5.4 equal to zero, we satisfy the perigee restriction of equation 3.19 and

solve for λe_o :

$$\lambda e_o \doteq 1 - \frac{9}{20}\theta_2^2 - \frac{3015 e_o^4 - 3672 e_o^3 - 10656 e_o^2 - 9936 e_o - 3492}{(1+e_o)^4} \frac{\theta_2^4}{16,800} \quad (5.11)$$

Here, we note that equation 5.11 would be much more complicated if a non-symmetrical thrust program were used. However, since we can see from the form of equation 5.6 that this will have no energy advantage, a symmetrical thrust program is chosen to simplify the results as much as possible.

Similarly, taking the derivative of equation 5.6 as indicated in equation 5.5, and using equation 5.11, we have the energy for one half of the cycle:

$$\frac{1}{2}\frac{\Delta E}{E g_o R_o} \doteq \frac{h_o^2}{1+e_o}\left\{\theta_2 - \frac{(1-2e_o)}{(1+e_o)}\frac{\theta_2^3}{6} + \frac{(7-13e_o+16e_o^2)}{(1+e_o)^2}\frac{\theta_2^5}{120} + \right.$$

$$\left. + \frac{2720 e_o^6 + 5190 e_o^5 + 3924 e_o^4 - 2504 e_o^3 - 1506 e_o^2 - 1134 e_o - 1111}{(1+e_o)^6}\frac{\theta_2^7}{50400}\right\}$$

(5.12)

To determine the advantage of this method of perigee restriction over that used by Casey, the energy equation given in reference (2) was expanded in series and integrated to the same number of terms as we have in equation 5.12. The two energies are identical to the last term, but if we compare the last term of the constant perigee energy expression:

$$+\frac{2720 e_o^6 + 5190 e_o^5 + 1125 e_o^4 - 2000 e_o^3 - 2550 e_o^2 - 4230 e_o - 2335}{(1+e_o)^6}\frac{\theta_2^7}{50400} \quad (5.13)$$

with the last term of equation 5.12, we see that for values of e_o

between zero and one, the constant perigee restriction yields less energy.

To find the changes of the elliptic orbit parameters u, h, and e, we note from equation 4.4 that

$$h_1(\theta_2) = \frac{1}{(A - A')} \frac{\Delta E}{\epsilon g_0 R_0} \tag{5.14}$$

where the energy is given by equation 5.12. Substituting the expressions for A and A' and neglecting the higher perturbations since ϵ is small, we have

$$\Delta h = \epsilon h_1(\theta_2) = \frac{h_0^3}{(1+e_0)^2} \frac{\Delta E}{g_0 R_0} \tag{5.15}$$

Now, from the perigee restriction, we may write equation 3.17 before and after the thrust cycle:

$$\frac{1 + e_0}{h_0^2} = \frac{1 + e_0 + \Delta e}{(h_0 + \Delta h)^2} \tag{5.16}$$

from which we solve for the change in eccentricity:

$$\Delta e = 2 \frac{h_0^2}{(1 + e_0)} \frac{\Delta E}{g_0 R_0} \tag{5.17}$$

using equation 5.15. To see if there is a shift of perigee, we assume that the equation for u, at the end of the thrust cycle, has the form of 2.12 :

$$u = \frac{[1 + (e_0 + \Delta e)\cos(\theta_2 - \Delta\theta_p)]}{(h_0 + \Delta h)^2} \tag{5.18}$$

where we have provided for a shift of the perigee with the term $\Delta\theta_p$. Then, using equations 2.12, 3.9, and 5.18, we can write

$$u_1 = \frac{1}{\epsilon}\left\{\frac{1+(e_0+\Delta e)\cos(\theta_2-\Delta\theta_p)-(1+e_0\cos\theta_2)\left(1+\frac{\Delta h}{h_0}\right)^2}{(h_0+\Delta h)^2}\right\} \tag{5.19}$$

and

$$u_1' = \frac{1}{\epsilon}\left\{\frac{-(e_0+\Delta e)\sin(\theta_2-\Delta\theta_p)+e_0\sin\theta_2\left(1+\frac{\Delta h}{h_0}\right)^2}{(h_0+\Delta h)^2}\right\} \tag{5.20}$$

Substituting these two equations into 4.3, with the expressions for A', B, and C, we obtain the result

$$(\cos(\theta_2-\Delta\theta_p)-\cos\theta_2)\cos\theta_2+(\sin(\theta_2-\Delta\theta_p)-\sin\theta_2)\sin\theta_2 = 0 + O(\epsilon) \tag{5.21}$$

Thus $\Delta\theta_p$ is at most of order ϵ. Using this result, we may evaluate the terms of order ϵ in equation 5.21, using 5.15 and 5.17, and we find that they are zero. Thus, there is no shift of the perigee within the magnitude of the first perturbation as would be expected with a symmetrical thrust program.

Before proceeding with the discontinuous thrust program, we shall obtain the approximate solution for the spiral orbit with thrust tangent to the velocity vector. For this case, ϕ is zero, and equation 3.8 becomes

$$h^2 = \frac{1}{u'' + u} \tag{5.22}$$

From equation 3.7 we have

$$h h' = \frac{\epsilon}{u^3}\left[1+\left(\frac{u'}{u}\right)^2\right]^{-\frac{1}{2}} \tag{5.23}$$

Taking twice of equation 5.23 equal to the derivative of 5.22 and expanding, we obtain

$$u u' = -2\epsilon\left[1 + 2\frac{u''}{u} - \frac{1}{2}\left(\frac{u'}{u}\right)^2 + \cdots\right] - u u''' \tag{5.24}$$

where we have assumed starting from a circular orbit with (u''/u) and (u'/u) much less than one. Integrating from 0 to θ we have:

$$u^2 = u_o^2 - 4\epsilon\theta - 2\int_0^\theta \left(u u''' + 4\epsilon\frac{u''}{u} - \epsilon\left(\frac{u'}{u}\right)^2 + \cdots\right) d\theta \tag{5.25}$$

To determine approximately the small quantities not yet integrated, let us take for the first approximate solution

$$u_a^2 = u_o^2 - 4\epsilon\theta \tag{5.26}$$

which is merely the first result from equation 5.25. Then, taking derivatives of equation 5.26, we have approximately:

$$\frac{u'}{u} = -\frac{2\epsilon}{u_a^2} \tag{5.27}$$

$$\frac{u''}{u} = -\frac{4\epsilon^2}{u_a^4} \tag{5.28}$$

$$u u''' = -\frac{24\epsilon^3}{u_a^4} \tag{5.29}$$

Substituting equations 5.27, 5.28, and 5.29 into 5.25, we obtain the result:

$$u^2 = u_a^2 + 22\epsilon^2\left[\frac{1}{u_a^2} - \frac{1}{u_o^2}\right] \tag{5.30}$$

where u_a is given by equation 5.26. We may now use the second term of equation 5.30 to choose values of ϵ and u_a, assuming u_o is given, to insure that our results lie within a certain accept-

able error. We should also note that (ϵ/U_a) must be much less than one if equation 5.30 is to have meaning.

To determine the time, we use part of equation 2.13 and substitute equations 5.22 and 5.30. Then, using equations 5.26 and 5.28, we obtain

$$\epsilon\tau = \int_0^\theta \frac{\epsilon\, d\theta}{h\, u^2} = \int_0^\theta \frac{\sqrt{u + u''}}{u^2}\, \epsilon\, d\theta = \int_0^\theta \left(1 + \frac{u''}{u}\right)^{\frac{1}{2}} (u)^{-3/2}\, \epsilon\, d\theta$$

$$\epsilon\tau = \int_0^\theta U_a^{-3/2}\left(1 - \frac{2\epsilon^2}{U_a^4} - \frac{33}{2}\epsilon^2\left[\frac{1}{U_a^4} - \frac{1}{U_o^2 U_a^2}\right] + \cdots\right)\epsilon\, d\theta$$

$$\epsilon\tau = \sqrt{U_o} - \sqrt{U_a} - \frac{\epsilon^2}{14}\left[\frac{37}{U_a^{7/2}} - \frac{77}{U_o^2 U_a^{3/2}} + \frac{40}{U_o^{7/2}}\right] \qquad (5.31)$$

Here, we see that for specific values of U_a and ϵ, we will have a larger error in equation 5.30 than in 5.31, since U_a is less than U_o and U_o is less than one.

To determine the energy input, we substitute equations 5.22, 5.27, 5.28, and 5.30 into 3.15, and take the difference between the final and initial values:

$$\frac{\Delta E}{g_o R_o} = \frac{1}{2}\left[(U_o - U_a) - \epsilon^2\left(\frac{3}{U_a^3} - \frac{11}{U_o^2 U_a}\right)\right] \qquad (5.32)$$

where again the error may be estimated by the second term. Of these three equations, 5.30, 5.31, and 5.32, equation 5.30 will have the largest error as U_a becomes small, and it should be used for

choosing values of ϵ and u_a .

We have reached the point where we must decide how to program the cycles for the restricted perigee. To do this, we shall choose a ratio of $(R_o/\Pi)=0.95$ to be the maximum value for u . For the case of the earth, this corresponds to an altitude of 208 miles and is adequate to insure negligible atmospheric effects.

If we form the quotient using equations 2. 13 and 5. 12,

$$\text{specific power} = \eta = \frac{1}{\epsilon g_o R_o} \frac{\Delta E}{\Delta \tau} \tag{5.33}$$

then expanding $\Delta \tau$ in series and performing the division

$$\eta = \frac{1+e_o}{h_o}\left\{1 - \frac{1}{1+e_o}\frac{\theta_z^2}{6} + \frac{21-13e_o}{(1+e_o)^2}\frac{\theta_z^4}{360} + \cdots\right\} \tag{5.34}$$

This equation is plotted in fig. 2 for various values of e_o , with corresponding values of $(E/g_o R_o)$ for $P_o = 0.95$, where P_o is defined as

$$P_o = \frac{1+e_o}{h_o^2} \tag{5.35}$$

Also plotted in fig. 2 is the specific power for the spiral orbit for the same values of $(E/g_o R_o)$. We can see that when the eccentricity is small, there is not much difference between the two methods near the perigee, but the discontinuous thrust program becomes inefficient if we proceed too far from the perigee. This is because the thrust must be directed away from the velocity vector to satisfy the perigee restriction. Thus, for a given energy increase per cycle when the eccentricity is small, ϵ must be larger with discontinuous thrust, since the thrust is not on for the full 2π radians. Therefore, $\Delta \Upsilon$ is larger, and the payload and structure to initial mass

ratio is reduced, as seen in equation 2.25. If we manipulate equations 2.13 and 3.15, we can obtain the time for 2π radians of travel in terms of the total energy:

$$\Delta T = \frac{\pi}{\sqrt{2}} \sqrt{\frac{R_o}{g_o}} \left[1 - \frac{E}{g_o R_o} \right]^{-\frac{3}{2}} \tag{5.36}$$

which holds for both elliptic and circular orbits. Now, if the mission is to be accomplished, both methods will have about the same number of cycles, where in the case of the continuous thrust method, cycle means 2π radians of travel. Thus, we want to impart about the same amount of energy during each corresponding cycle.

We can also see in fig. 2 that for larger values of eccentricity, say 0.75, the specific power is much greater for the elliptic orbit. In this case, for a given energy increase per cycle, $\Delta\Upsilon$ for the spiral orbit will be greater.

VI. APPLICATION TO MULTIPLE CYCLES

To find the total energy input, total time, and γ , we must make a summation of the results for each cycle. However, since we are assuming a large number of cycles, with small changes for each cycle, we may write integrals for the summation operation and obtain our results by integration. The thrusting time, Δt , is taken as constant for all of the cycles, and the variable of integration is de .

We must first express the cutoff angle, θ_2 , in terms of e because it must be varied slightly to keep the thrusting time constant. Expanding equation 2.13 in series, we have, for the symmetrical orbit:

$$\Delta t\sqrt{\frac{g_o}{R_o}} = \frac{2\,\theta_2}{P_o^{3/2}\sqrt{1+e}}\left\{1 + \frac{1}{3}\,\frac{e}{(1+e)}\,\theta_2^2 - \frac{e(1-8e)}{60(1+e)^2}\,\theta_2^4 + \cdots\right\} \tag{6.1}$$

Then, if we let

$$k = \frac{P_o^{3/2}}{2}\sqrt{\frac{g_o}{R_o}}\,\Delta t = \frac{P_o^{3/2}}{2}\,\Delta\tau \tag{6.2}$$

we can write approximately

$$\theta_2 = k\sqrt{1+e}\left\{1 - \frac{1}{3}e k^2 + \frac{2}{9}\,\frac{e^2}{(1+e)}\,k^4\right\} \tag{6.3}$$

where k is the initial value of θ_2 . Care must be used in choosing k so that θ_2 is less than one at $e = 1$; otherwise, the series expansions will not be good approximations.

Substituting equation 6.3 into 5.17, using 5.12, and replacing the finite "delta" quantities with differentials, we obtain

$$\epsilon\, di = \frac{P_o^2}{4k}\left\{\left(1+\frac{k^2}{6}\right)\frac{1}{\sqrt{1+e}} + \frac{72e^3-29e^2-32e-11}{360(1+e)^{3/2}}\,k^4\right\}de \qquad (6.4)$$

The term di is in effect one, that is, one cycle, and (de/ϵ) is the small change of eccentricity during that one cycle. To perform the integration, we shall take ϵ as constant to obtain:

$$\epsilon N = \frac{P_o^2}{4k}\left\{\left(2+\frac{k^2}{3}\right)\sqrt{1+e} + \frac{3821+1828e-577e^2+216e^3}{2700\sqrt{1+e}}\,k^4\right\}\Bigg]_{e_i}^{e_f} \qquad (6.5)$$

where e_i is the initial eccentricity and e_f is the final. N is the number of thrust cycles, and the total thrusting time is given by:

$$\epsilon\,\tau_T = \epsilon N \Delta\tau \qquad (6.6)$$

We find the energy input very simply by replacing the finite "delta" quantities in equation 5.17 with differentials, using 5.35, and integrating to obtain

$$E_f - E_i = \frac{g_o R_o}{2} P_o\,(e_f - e_i) \qquad (6.7)$$

Then, using equations 2.22, 3.1, and 6.6,

$$\gamma^2 = \frac{K}{2} g_o^2 \epsilon^2 N \Delta t = \frac{K}{2} g_o^2 \epsilon^2 \sqrt{\frac{R_o}{g_o}}\, N \Delta\tau \qquad (6.8)$$

To find the total travel time for 2π radians, we write equation 5.36 in dimensionless form, replacing the finite "delta" quantities with differentials; then substituting 3.15 and 5.35, we have

$$d\tau = \frac{2\pi}{P_o^{3/2}} \frac{di}{(1-e)^{3/2}} \tag{6.9}$$

where again, di is in effect one. Substituting equation 6.4 for di and neglecting "end effects" in the integration, we have

$$\epsilon\tau = \frac{2\pi\sqrt{P_o}}{4k}\left\{\left(1+\frac{k^2}{6}\right)\frac{(1+e)}{\sqrt{1-e^2}} + \frac{k^4}{360}\left(29\sin^{-1}e + \frac{112-40e-72e^2}{\sqrt{1-e^2}}\right)\right\}\Bigg]_{e_i}^{e_f} \tag{6.10}$$

The "end effects" we mention are the initial cutoff angle and the final cutoff angle not being equal so that we have a small amount of time neglected for each cycle due to the shift of the cutoff angle; the final energy is added before starting the final coasting orbit; and the initial energy is added during the latter part of the initial coasting orbit. This might be clearer if we note that the integration gives us

$$\tau = \Delta\tau_{e_i} + \Delta\tau_{e_{i+1}} + \cdots + \Delta\tau_{e_{f-1}} + \Delta\tau_{e_f} \tag{6.11}$$

where $\Delta\tau_{e_j}$ is the travel time for 2π radians at eccentricity e_j, whereas the true mission time would be given by:

$$\tau = \Delta\tau_i + \Delta\tau_{e_{i+1}} + d\tau_{i+1} + \Delta\tau_{e_{i+2}} + d\tau_{i+2} + \cdots + d\tau_{f-2} + \Delta\tau_{e_{f-1}} + d\tau_{f-1} \tag{6.12}$$

where $\Delta\tau_i$ is the thrust time for the initial cycle, $e_j = e_{j-1} + \Delta e_j$, and $d\tau_j$ is the cutoff angle shift time. If the number of cycles is not large, then equation 6.10 should be corrected for "end effects" by adding

$$\tau_{CORR} = \Delta\tau_i\left(1 + \frac{\theta_{2f}-k}{\theta_{2f}+k}\right) - \Delta\tau_{e_i} - \Delta\tau_{e_f} \tag{6.13}$$

where k is the initial cutoff angle given by equation 6.2, θ_{2f} is obtained by evaluating equation 6.3 at $e = e_f$. The cutoff angle shift time correction is an average value obtained by using the first term of equation 6.1. The other corrections follow simply by comparing equations 6.11 and 6.12.

VII. RESTRICTED PERIGEE CYCLE WITH BATTERY STORAGE

We must still consider the possibility of different values of specific power plant mass, K , for the two systems, since we have the capability of storing energy with the discontinuous method while coasting if we provide batteries. The energy generated by the power supply while coasting could be stored in batteries and then used in addition to the power supply while thrusting.

To study this possibility, we shall first assume that a battery pack is included with the power supply and that other conditions of the thrust program are unchanged; later the battery will be considered as part of the payload. We shall further assume that during the early part of the program, when e is small, that the coasting time is not sufficient to fully charge the batteries, but as e increases, a point is reached where the batteries do become fully charged, and are fully recharged during each subsequent coasting period. We have equation 2.18, evaluated at the "burnout condition":

$$\frac{M_o}{M_b} = 1 + \frac{M_o}{M_w} \frac{K}{2} \int_0^{T_b} \frac{M_w}{K} \frac{a^2}{P} \, dt \tag{7.1}$$

where we have multiplied and divided by (K/M_w), and must now consider P as a function of time. When the batteries are not fully charged, the power is given by

$$P = \frac{M_g}{K} \frac{\Delta \tau_{CYCLE}}{\Delta \tau} \tag{7.2}$$

neglecting battery losses while charging, where M_g is the mass of

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neglecting battery losses while charging, where M_g is the mass of

the electrical generating unit, K is the specific mass of the generating unit, $\Delta\tau$ is the constant thrusting time per cycle, and $\Delta\tau_{cycle}$ is the travel time for 2π radians given by equation 5.36. The batteries will be fully charged when

$$\frac{M_{batt.}}{K_{batt.}} = \frac{M_g}{K}\left(\Delta\tau_{orbit} - \Delta\tau\right) \tag{7.3}$$

or

$$\Delta\tau_{orbit} \geq \Delta\tau + \frac{M_{batt.}}{M_g}\ \frac{K}{K_{batt.}} \tag{7.4}$$

where $M_{batt.}$ and $K_{batt.}$ are the mass and the specific mass of the battery, respectively. From equation 7.1, we define γ^2, corresponding to 2.22:

$$\gamma^2 = \frac{K}{2}\int_0^T \frac{M_w}{K}\ \frac{a^2}{P}\,dt = \frac{1}{2}\int_0^T \frac{M_w}{P}\,a^2\,dt \tag{7.5}$$

But since ϵ is constant, and $M_w = M_g + M_{batt.}$:

$$\gamma^2 = \frac{\epsilon^2 g_0^2}{2}\int_0^T \frac{M_g\left(1 + \frac{M_{batt.}}{M_g}\right) f(t)}{P}\,dt \tag{7.6}$$

where $f(t)$ is introduced to remind us that the integrand is zero when not thrusting. Using equation 7.4 in 7.2, we can write the maximum power available when the batteries are fully charged:

$$P_{MAX} = \frac{M_g}{K}\left(1 + \frac{M_{batt.}}{M_g}\ \frac{K}{K_{batt.}}\ \frac{1}{\Delta\tau}\right) \tag{7.7}$$

Using equations 7.2 and 7.7 in 7.6 :

$$\gamma^2 = \frac{\epsilon^2 g_o^2}{2} K\left(1 + \frac{M_{batt.}}{M_g}\right)\left[\int_0^{t_{crit.}} \frac{f(t)\,\Delta\tau}{\Delta\tau_{ORBIT}}\,dt + \int_{t_{CRIT}}^{T} \frac{f(t)}{\left(1 + \frac{1}{\Delta\tau}\frac{M_{batt.}}{M_g}\frac{K}{K_{batt.}}\right)}\,dt\right] \tag{7.8}$$

where $t_{crit.}$ is the time at which the batteries become fully charged. Now, the integrals in equation 7.8 may be treated as summations since the thrust is on for the constant time $\Delta\tau$ during each cycle:

$$\gamma^2 = \frac{\epsilon^2 g_o^2}{2} K\left(1 + \frac{M_{batt.}}{M_g}\right)\sqrt{\frac{R_o}{g_o}}\left[(\Delta\tau)^2 \sum_{n=1}^{N_{CRIT}} \frac{1}{\Delta\tau_{orbit_n}} + \frac{(N - N_{CRIT})\,\Delta\tau}{\left(1 + \frac{1}{\Delta\tau}\frac{M_{batt.}}{M_g}\frac{K}{K_{batt.}}\right)}\right] \tag{7.9}$$

where $N_{crit.}$ is the cycle number after which the batteries are fully charged. Using equations 6.9 and 7.4, we can solve for the eccentricity when the batteries become fully charged:

$$e_c = 1 - \frac{1}{P_o}\left[\frac{2\pi}{\Delta\tau + \frac{M_{batt.}}{M_g}\frac{K}{K_{batt.}}}\right] \tag{7.10}$$

Then, using equations 6.4 and 6.9 :

$$\sum_{n=1}^{N_{crit}} \frac{1}{\Delta\tau_{orbit_n}} = \sum_{n=1}^{N_{crit}} \left(\frac{di}{d\tau}\right)_n = \int_{n=1}^{N_{crit}} \left(\frac{di}{d\tau}\right) di =$$

$$= \frac{P_o^{7/2}}{8\pi\epsilon k}\int_0^{e_c} (1-e)^{3/2}\left\{\left(1 + \frac{k^2}{6}\right)\frac{1}{\sqrt{1+e}} + \frac{72e^3 - 29e^2 - 32e - 11}{360(1+e)^{3/2}} k^4\right\} de \tag{7.11}$$

Integrating this equation and determining $N - N_{crit.}$ from equation

6.5, using 7.10, equation 7.9 becomes:

$$\gamma^2 = \frac{\epsilon^2 g_o^2}{2} K(1+\mu)\sqrt{\frac{R_o}{g_o}}\,\frac{P_o^2}{4\epsilon k}\left\{(\Delta\tau)^2\frac{P_o^{3/2}}{2\pi}\left[\left(1+\frac{k^2}{6}\right)\left(\frac{3}{2}\sin^{-1}e_c+\left(2-\frac{e_c}{2}\right)\sqrt{1-e_c^2}-2\right)+\right.\right.$$

$$\left.+\left(\frac{108e_c^5-290e_c^4+1083e_c^3-2404e_c^2-3111e_c+4814}{6\sqrt{1-e_c^2}}+\frac{1015}{2}\sin^{-1}e_c-\frac{2407}{3}\right)\frac{k^4}{360}\right]+$$

$$\left.+\frac{\Delta\tau}{(1+\alpha)}\left[\left(2+\frac{k^2}{3}\right)\sqrt{1+e}+\frac{3821+1828e-577e^2+216e^3}{2700\sqrt{1+e}}k^4\right]_{e_c}^{e_f}\right\}$$

(7.12)

where

$$1+\mu = 1+\frac{M_{batt.}}{M_g} \qquad (7.13)$$

and

$$1+\alpha = 1+\frac{1}{\Delta\tau}\,\frac{M_{batt.}}{M_g}\,\frac{K}{K_{batt.}} \qquad (7.14)$$

Now if we form the ratio of equation 2.22 for the spiral orbit and equation 7.12, multiplied by the factor $1+\mu$, we have

$$FR_{\gamma 2} = (1+\mu)\frac{\gamma_s^2}{\gamma_d^2} = \left\{\frac{4k\,(\epsilon\tau)_s}{(\epsilon_d/\epsilon_s)\,P_o^2}\right\} \div \left\{(\Delta\tau)^2 \frac{P_o^{3/2}}{2\pi}\left[\left(1+\frac{k^2}{6}\right)\left(\frac{3}{2}\sin^{-1}e_c + \right.\right.\right.$$

$$\left. + \left(2-\frac{e_c}{2}\right)\sqrt{1-e_c^2} - 2\right) + \left(\frac{108e_c^5 - 290e_c^4 + 1083e_c^3 - 2404e_c^2 - 3111e_c + 4814}{6\sqrt{1-e_c^2}} + \right.$$

$$\left. \left. + \frac{1015}{2}\sin^{-1}e_c - \frac{2407}{3}\right)\frac{k^4}{360}\right] + \frac{\Delta\tau}{(1+\alpha)}\left[\left(2+\frac{k^2}{3}\right)\sqrt{1+e} + \right.$$

$$\left.\left. + \frac{3821 + 1828e - 577e^2 + 216e^3}{2700\sqrt{1+e}}\,k^4\right]_{e_c}^{e_f}\right\} \qquad (7.15)$$

When $FR_{\gamma 2}$ is less than one, the spiral method of energy addition must have a γ less than the γ for the restricted perigee method, and by equation 2.25 the spiral method must have a greater payload. If $FR_{\gamma 2}$ is greater than one, then there may be a possibility of carrying a greater payload by using the restricted perigee method. This will depend on the ratio of $K/K_{batt.}$ that can be obtained.

VIII. COMPARISON OF RESULTS

Now to compare the results we have obtained, equation 5.31, the spiral thrust time, is plotted against the energy added, equation 5.32, in fig. 3. Using a value of $k = 0.8$, which corresponds to a maximum $\theta_2 = 0.944$ by equation 6.3, the restricted perigee thrust time is computed using equations 6.2, 6.5, and 6.6 for values of eccentricity from zero to 0.95, with $P_o = 0.95$. The corresponding increase of energy is computed using equation 6.7, and these results are plotted in fig. 3. Since the total time must be equal for either method at a specified energy addition, we may compute equations 5.31 and 6.10 at specific energy levels and take their ratio to plot $(\epsilon_d / \epsilon_s)$ in fig. 3. This is the ratio of acceleration for the discontinuous to the spiral that is necessary to accomplish the mission.

It seems clear from fig. 3, in view of the acceleration ratios required for the mission and equations 2.13, 2.22, and 6.8, that γ^2 will be greater for the discontinuous method if K is the same for both methods and starting from the circular orbit with $P_o = 0.95$. For these conditions, the discontinuous thrust method will provide less payload. Further, if we decrease k, the initial cutoff angle, we see from equation 6.10 that ϵ must be increased for the same total time; thus, γ^2 will be even larger for the discontinuous method, resulting in even less payload.

Using equation 7.10 and fig. 3, we have evaluated equation 7.15 for three different final energies, expressed as eccentricity, for

various values of α. These results are plotted in fig. 4.

From reference (3), we find that a typical value of K would be about 150 pounds per kilowatt. This value will decrease as space power plant technology progresses. In reference (3), we also find $K_{batt.}$ of 75 pounds per kilowatt-hour for nickel-cadmium batteries, which have very reliable charge and discharge characteristics, and 14 pounds per kilowatt-hour for silver-zinc batteries, which are not as reliable for a large number of recharging cycles. From equation 7.3, we see that both values of $K_{batt.}$ must be divided by $3600(g_o/R_o)^{1/2}$ to have the proper non-dimensional time units. Thus, we have the values for $K_{batt.}$ of 16.8 for nickel-cadmium and 3.13 for silver-zinc.

We cannot improve over the spiral thrust program with either battery supply if the final energy must correspond to an eccentricity of 0.95, since in fig. 4 the maximum $FR_{\gamma 2}$ is 1.006.

Using equations 5.35 and 6.9, we compute the non-dimensional thrust time for one cycle, corresponding to an initial cutoff angle of 0.8 and $e_o = 0$:

$$\Delta \tau = \frac{(2)(0.8)}{2\pi} \frac{d\tau}{di} = \frac{1.6}{(0.95)^{3/2}} = 1.728 \tag{8.1}$$

Using this and equation 7.14, we compute the values of μ for the nickel-cadmium battery, for $\alpha = 7.5$ and $\alpha = 10.0$, respectively:

$$\mu = 1.45 \quad , \quad \mu = 1.93 \tag{8.2}$$

and for the silver-zinc battery:

$$\mu = 0.27 \quad , \quad \mu = 0.36 \tag{8.3}$$

From fig. 4, we read $FR_{\gamma^2} = 1.21$ for the final eccentricity of 0.4 and $\alpha = 7.5$. But from equations 7.15, 8.2, and 8.3, the values of μ for either battery will require γ_d^2 to be greater than γ_s^2; again, we find no advantage in the restricted perigee method of energy addition. However, for the final eccentricity of 0.7 we read from fig. 4 FR_{γ^2} of 1.29 and 1.39 for α's of 7.5 and 10.0, respectively. From equations 7.15 and 8.3, we compute for this condition:

$$\frac{\gamma_s}{\gamma_d} = 1.008 \quad , \quad \frac{\gamma_s}{\gamma_d} = 1.010 \tag{8.4}$$

for the two values of α, respectively. From equation 2.25, we can see that in this case the restricted perigee method has a slightly larger payload than the spiral method for this particular final energy. To see this result more clearly, we form the ratio of equation 2.25 for the two methods using 8.4:

$$\frac{(M_1 + M_s)_d}{(M_1 + M_s)_s} = \frac{(1 - \gamma_d)^2}{(1 - 1.01\,\gamma_d)^2} \tag{8.5}$$

for $\alpha = 10.0$. Now, clearly, the numerator of 8.5 is greater than the denominator, so the ratio must be greater than one.

Although we have found that the restricted perigee method of energy addition, using silver-zinc batteries, yields a slightly larger payload than the spiral method for certain values of final energy, we still have the problem of battery reliability. There is no question of the recharging reliability of the nickel-cadmium battery, but we found the specific mass, $K_{batt.}$, to be too high to be useful. The silver-zinc battery could be used for a slight payload advantage, be-

cause of its lower specific mass, but its recharging reliability must first be improved.

We must finally compare the two methods of energy addition when reliable, rechargeable batteries are part of the payload but are available to be used in the thrust program. In this case, $\mu = 0$ and we see immediately from fig. 4 that there are certain values of and final energy where the restricted perigee has a decided advantage over the spiral method.

As an example, let us determine the advantage where a nickel-cadmium battery, $K_{batt.} = 16.8$, is part of the payload. Using equations 7.14 and 8.1, we compute values for $(M_{batt.}/M_g)$ for $\alpha = 7.5$ and $\alpha = 10.0$, respectively:

$$\frac{M_{batt.}}{M_g} = 1.45 \qquad\qquad \frac{M_{batt.}}{M_g} = 1.93 \tag{8.6}$$

Even though the ratio of battery to power supply mass is rather large, it is not unrealistic since we could easily have a satellite where both battery and power supply are but a small fraction of the initial mass. From fig. 4 we read $FR_{\gamma 2} = 1.21$ for the final eccentricity of 0.4 and $\alpha = 7.5$, and since $\mu = 0$:

$$\frac{\gamma_s}{\gamma_d} = 1.10 \tag{8.7}$$

Forming the ratio of equation 2.25 for the two methods, using 8.7 :

$$\frac{(M_1 + M_s)_d}{(M_1 + M_s)_s} = \frac{(1 - \gamma_d)^2}{(1 - 1.10\gamma_d)^2} \tag{3.8}$$

Similarly, from fig. 4 we read $FR_{\gamma 2}$ of 1.29 and 1.39 for the final eccentricity of 0.7 and α's of 7.5 and 10.0, respectively.

Since $\mu = 0$, we have immediately:

$$\frac{\gamma_s}{\gamma_d} = 1.136 \quad , \quad \frac{\gamma_s}{\gamma_d} = 1.179 \tag{8.9}$$

for the two values of α respectively. Again forming the ratio of equation 2.25 for the two methods, using 8.9, we have

$$\frac{(M_1 + M_s)_d}{(M_1 + M_s)_s} = \frac{(1-\gamma_d)^2}{(1-1.136\,\gamma_d)^2} \quad , \quad \frac{(M_1 + M_s)_d}{(M_1 + M_s)_s} = \frac{(1-\gamma_d)^2}{(1-1.179\,\gamma_d)^2} \tag{8.10}$$

for the two values of $\alpha = 7.5$ and $\alpha = 10.0$, respectively. The ratios in equations 8.8 and 8.10 will depend on the value determined for γ_d, computed from equation 7.12, but they do show a definite advantage for the restricted perigee method of energy addition.

For example, suppose the value for γ_d is 0.25, that is, let 25 per cent of M_0 be propellant. Substituting in equation 8.10, we compute ratios of 1.097 and 1.131 for the two values of α, respectively. Now, this is about a 10 per cent gain in payload and structure mass by using the discontinuous rather than the continuous method of thrust programming, where the batteries are part of the payload.

It seems rather surprising that we have found no advantage for the restricted perigee method of energy addition for the final eccentricity of 0.95, where we found a definite advantage for the final eccentricity of 0.7. There are two reasons for this result. First, we can see from equation 6.9 that the travel time for 2π radians increases as the eccentricity increases. Thus, if we thrust for the same amount of time during each cycle, there is a greater percentage

of coasting time as the eccentricity increases. This requires an increase of $(\epsilon_d / \epsilon_s)$, as shown in fig. 3, to accomplish the mission, and in turn, we find a decrease in the ratio of (γ_s / γ_d) so that there is no advantage when e_f reaches 0.95 . Second, by restricting the cutoff angle to one radian or less, we do not take full advantage of the greater velocities provided by the elliptic orbit as the eccentricity approaches one. This can be seen clearly in fig. 2 where, at a cutoff angle of one radian and $e = 0.95$, the specific power of the restricted perigee is many times greater than the spiral method of energy addition. To obtain the full advantage of this much greater specific power, we must go to larger cutoff angles.

A study of the restricted perigee method of energy addition at larger cutoff angles would require the use of numerical methods. This would be rather involved, using equation 4.13 to evaluate λe for various values of θ_2 and e , then using this result for a corresponding evaluation of the change in energy, equation 4.12. After tabulating these "cycle" results, summations of time and energy could be made with a corresponding calculation of γ^2 for various values of α . By making such a numerical study, we can clearly define the area where the restricted perigee method, with batteries, has an advantage over the spiral method of energy addition. We will also be able to determine if there is any area where an advantage exists without the use of batteries.

IX. CONCLUSION

We have found that the discontinuous method of energy addition, using a restricted perigee, has no payload and structure mass ratio advantage over the spiral method when a specified energy is imparted in a specified time and the cutoff angle is restricted to one radian or less. There is the possibility of a very small advantage, for a certain range of final energies, if silver-zinc batteries are included in the power supply to store energy during the coasting periods, providing their reliability can be improved to an acceptable level. On the other hand, a moderate decrease of the power-plant specific mass will rule out any advantage of including batteries in the power supply.

In the case where batteries are included in the payload, say, to periodically energize a transmitter after the satellite has been placed in final orbit, we can definitely increase the payload and structure mass ratio for certain values of final energy by using these batteries in the restricted perigee thrust program. This does not apply in general, obviously, since the mass fraction of the batteries required in the payload must be compatible with the mass fraction of batteries required for the thrust program to accomplish the mission. When we do have a compatible battery requirement, then we can increase the payload and structure mass ratio by using the restricted perigee method of energy addition.

Since we are restricting our investigation to low thrust devices, a numerical study of the restricted perigee method of energy addition at cutoff angles greater than one radian could be made using

the perturbation equations developed earlier. Such a study would follow the procedure briefly outlined in the last paragraph of the preceding section, and is necessary to clearly define the area where the restricted perigee method can be used to advantage.

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3. Riley, F. E., and Sailer, J. D., Space Systems Engineering, McGraw-Hill Book Company, Inc., New York (1962).

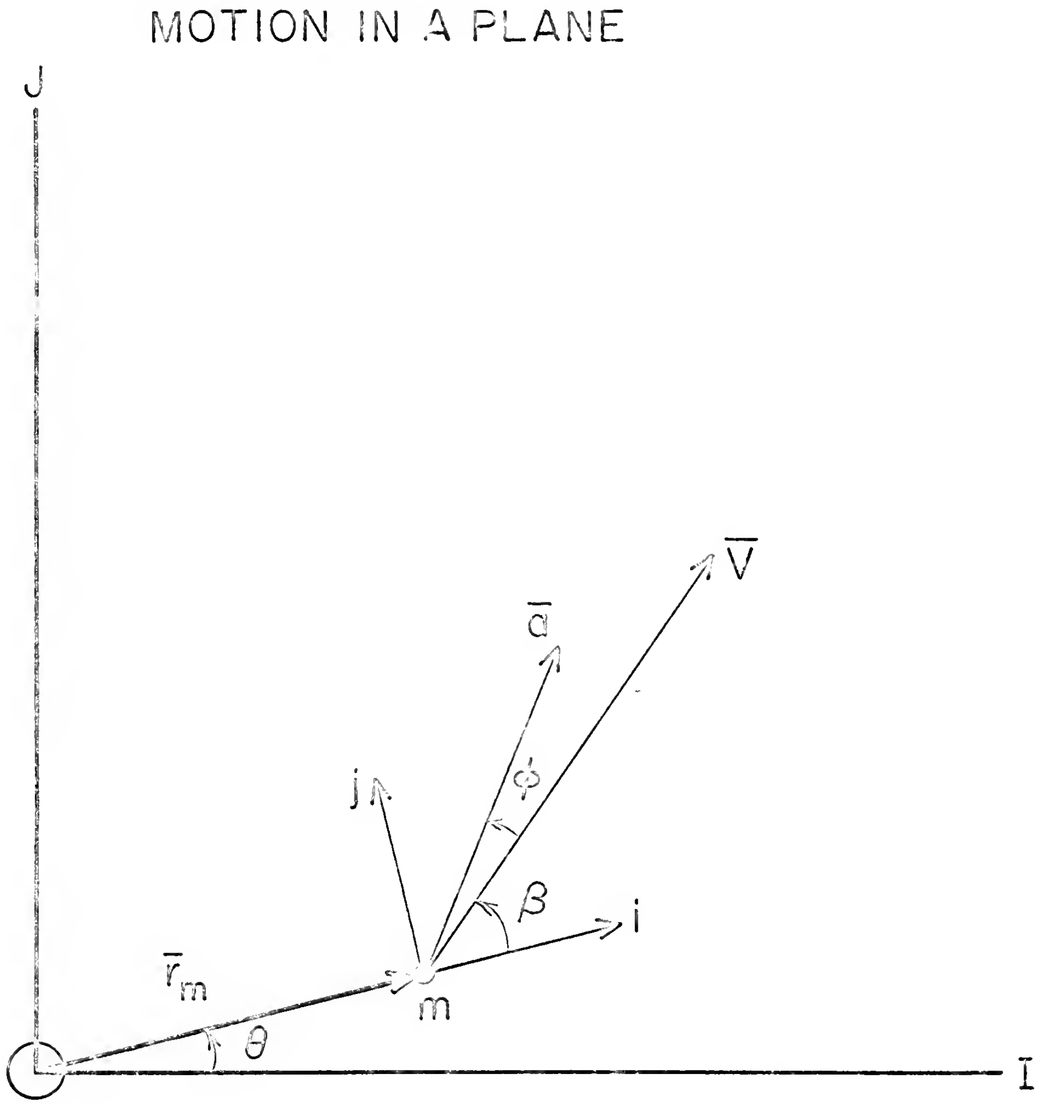

FIGURE 1

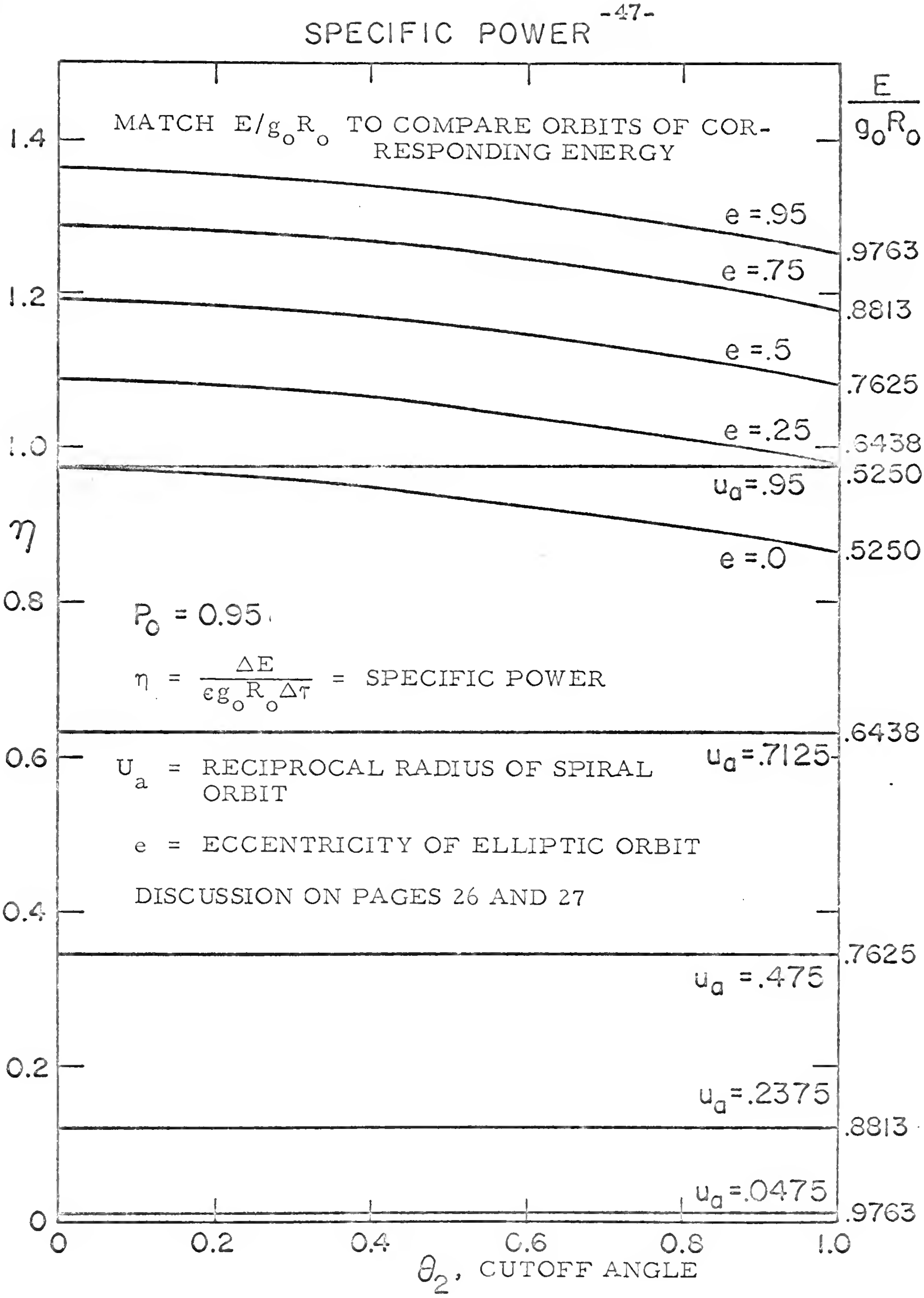
SPECIFIC POWER
MATCH E/g_o R_o TO COMPARE ORBITS OF CORRESPONDING ENERGY
e =.95
e =.75
e =.5
e =.25
u_a=.95
e =.0
P_o = 0.95
η = ΔE / εg_o R_o Δτ = SPECIFIC POWER
u_a=.7125
U_a = RECIPROCAL RADIUS OF SPIRAL ORBIT
e = ECCENTRICITY OF ELLIPTIC ORBIT
DISCUSSION ON PAGES 26 AND 27
u_a =.475
u_a=.2375
u_a=.0475
η
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
θ_2, CUTOFF ANGLE
E/g_o R_o
.9763
.8813
.7625
.6438
.5250
.5250
.6438
.7625
.8813
.9763

FIGURE 2

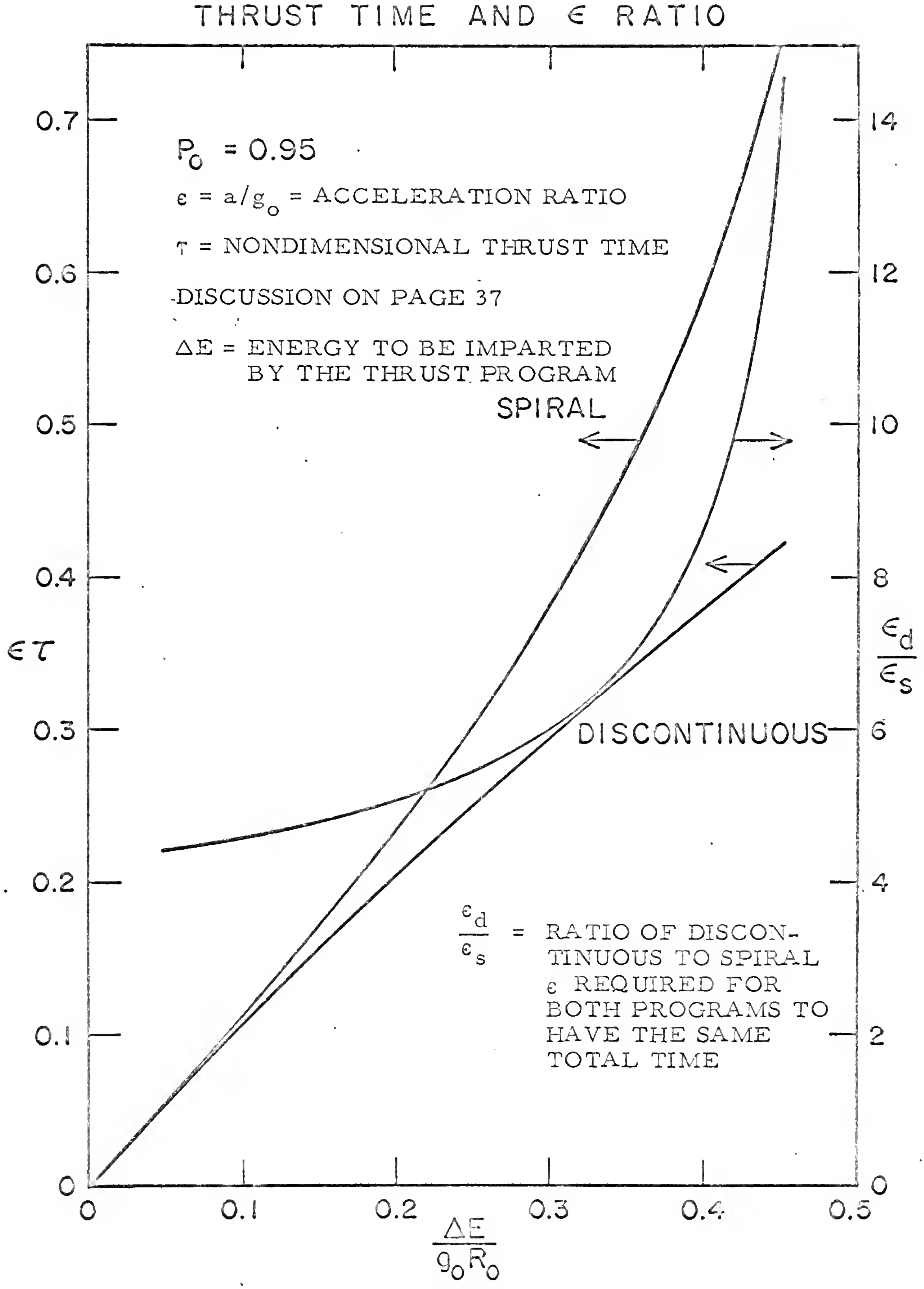
THRUST TIME AND ∈ RATIO
$P_0 = 0.95$
$\epsilon = a/g_o$ = ACCELERATION RATIO
τ = NONDIMENSIONAL THRUST TIME
DISCUSSION ON PAGE 37
ΔE = ENERGY TO BE IMPARTED BY THE THRUST PROGRAM
SPIRAL
DISCONTINUOUS
$\frac{\epsilon_d}{\epsilon_s}$ = RATIO OF DISCONTINUOUS TO SPIRAL ϵ REQUIRED FOR BOTH PROGRAMS TO HAVE THE SAME TOTAL TIME
$\epsilon\tau$
$\frac{\epsilon_d}{\epsilon_s}$
$\frac{\Delta E}{g_0 R_0}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
2
4
6
8
10
12
14

FIGURE 3

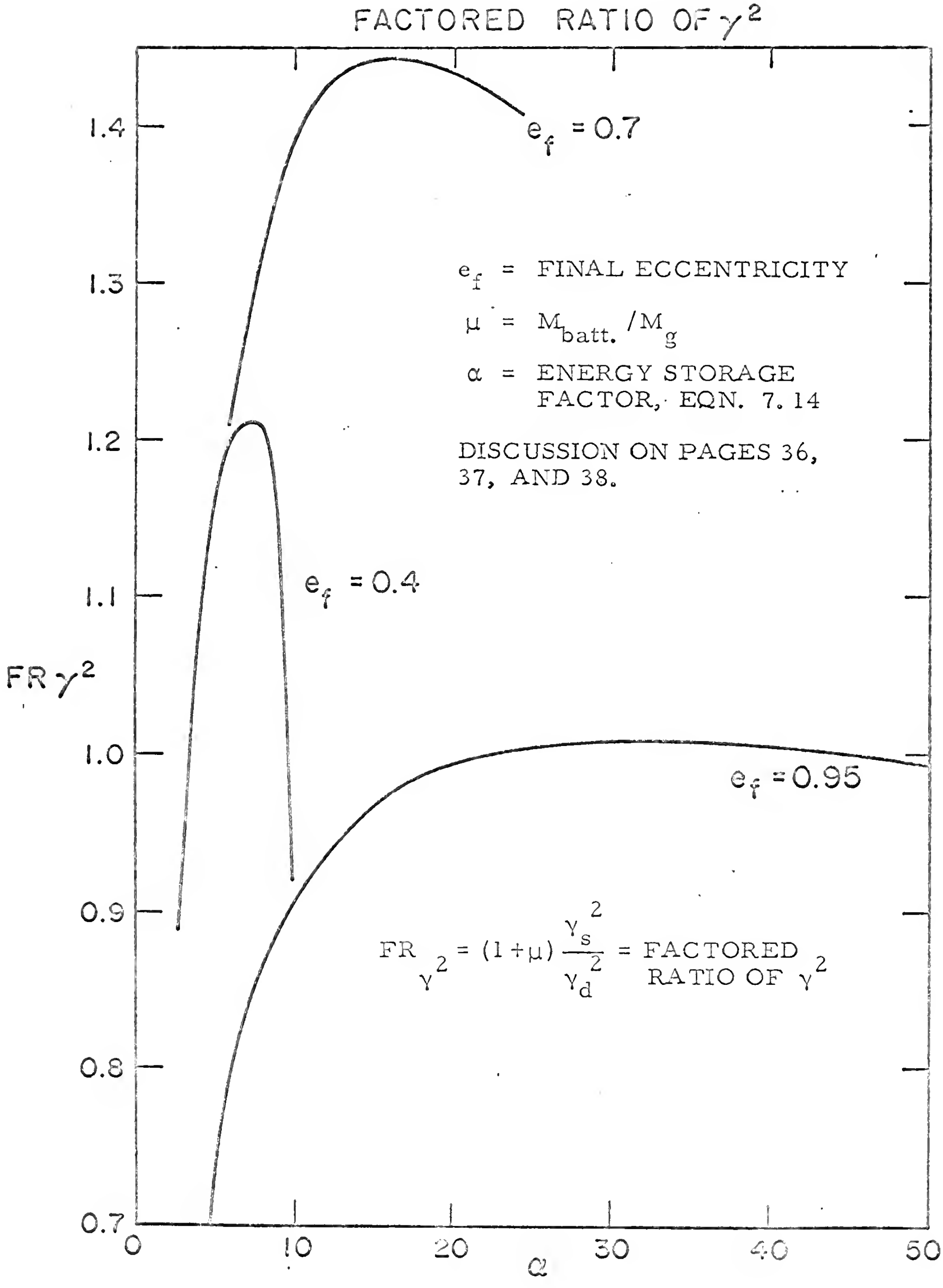
FACTORED RATIO OF γ^2
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
FR γ^2
0
10
20
30
40
50
α
e_f = 0.7
e_f = 0.4
e_f = 0.95
e_f = FINAL ECCENTRICITY
μ = $M_{batt.}/M_g$
α = ENERGY STORAGE FACTOR, EQN. 7.14
DISCUSSION ON PAGES 36, 37, AND 38.
$FR_{\gamma^2} = (1+\mu)\frac{\gamma_s^2}{\gamma_d^2}$ = FACTORED RATIO OF γ^2

FIGURE 4